EE120 - Fall'19 - Lecture 19 Notes[1]

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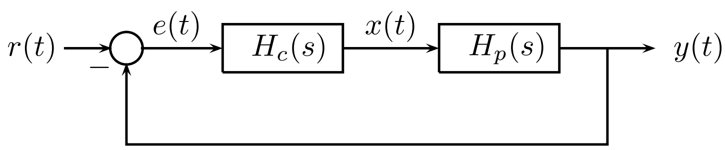
12 November 2019

Feedback Control

Chapter 11 *in* Oppenheim & Willsky



$r(t)$: reference signal to be tracked by $y(t)$

$H_c(s)$: controller; $H_p(s)$: system to be controlled ("plant")

Closed-loop transfer function:

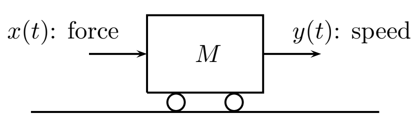
$$H(s) = \frac{Y(s)}{R(s)} = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}$$

Constant-gain control: $H_c(s) = K$

$$H(s) = \frac{KH_p(s)}{1 + KH_p(s)}$$

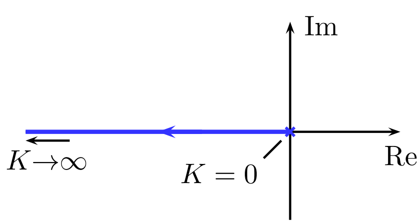
Closed-loop poles: roots of $\boxed{1 + KH_p(s) = 0}$

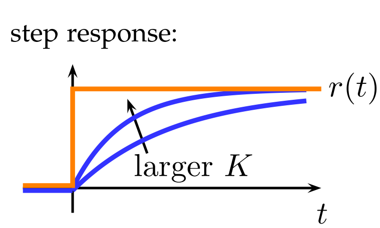
Example 1 (Speed Control)



$$H_p(s) = \frac{1}{Ms} \longrightarrow \text{ open-loop pole: } s = 0$$

Closed-loop pole: $1 + K\frac{1}{Ms} = 0 \Longrightarrow s = -\frac{K}{M}$





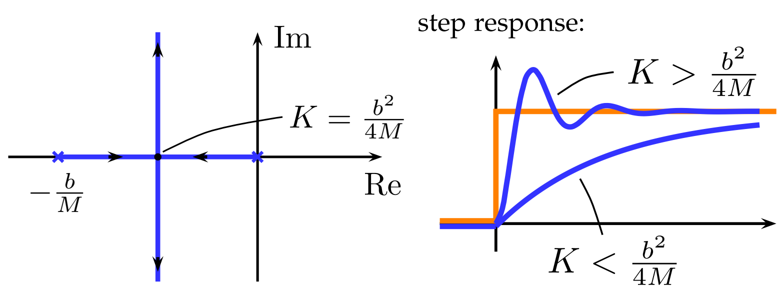
Example 2 (Position Control) $y(t)$: position

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} = x(t) \quad H_p(s) = \frac{1}{Ms^2 + bs} = \frac{1}{s(Ms+b)}$$

Open-loop poles: $s = 0, \frac{-b}{M}$
Closed-loop poles:

$$1 + \frac{K}{s(Ms+b)} = 0 \Longrightarrow \qquad Ms^2 + bs + K = 0$$

$$s = \frac{-b \mp \sqrt{b^2 - 4KM}}{2M}$$



Root-Locus Analysis

Section 11.3 *in* Oppenheim & Willsky

How do the roots of

$$\boxed{1 + KH(s) = 0}$$

move as K is increased from $K = 0$ to $K = +\infty$?

If a point $s_0 \in \mathcal{C}$ is on the root locus, then $H(s_0) = \frac{-1}{K}$ for some $K > 0$, therefore $\measuredangle H(s_0) = \pi$. The rules for sketching the root locus below are derived from this property.

Rules for sketching the root locus:
Let

$$\begin{aligned} H(s) &= \frac{s^m + b_{m-1}s^{m-1} + ... + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_0} \quad m \leq n \\ &= \frac{\prod_{k=1}^{m}(s - \beta_k)}{\prod_{k=1}^{n}(s - \alpha_k)} \qquad \begin{array}{l} \beta_k: \text{ zeros } k = 1, ..., m \\ \alpha_k: \text{ poles } k = 1, ..., n \end{array} \end{aligned}$$

1) As $K \to 0$, the roots converge to the poles of $H(s)$:

$$H(s) = -\frac{1}{K} \to \infty$$

Since there are n poles, the root locus has n branches, each starting at a pole of $H(s)$.

2) As $K \to \infty$, m branches approach the zeros of $H(s)$. If $m < n$, then $n - m$ branches approach infinity following asymptotes centered at:

$$\boxed{\frac{\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{m} \beta_k}{n-m}}$$

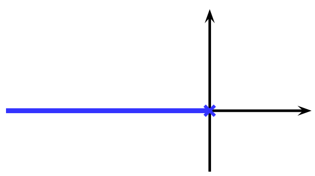
with angles:

$$\boxed{\frac{180^o + (i-1)360^o}{n-m} \quad i = 1, 2, ..., n-m.}$$

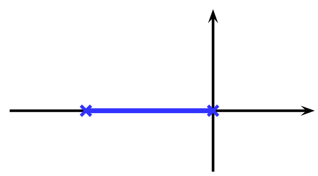
Example 2 above: $n - m = 2$, poles: $0, -b/M$
with center $= \frac{-b}{2M}$, and angles $= 90^o$, -90^o

3) Parts of the real line that lie to the left of an odd number of real poles and zeros of $H(s)$ are on the root locus.

Example 1 above:



Example 2:

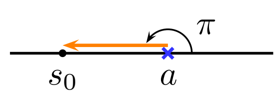


Proof of Property 3:

$$\measuredangle H(s_0) = \sum_{k=1}^{m} \measuredangle(s_0 - \beta_k) - \sum_{k=1}^{n} \measuredangle(s_0 - \alpha_k)$$

If s_0 is on the real line:

$$\measuredangle(s_0 - a) = \begin{cases} \pi & \text{if} \quad s_0 < a \\ 0 & \text{if} \quad s_0 > a \end{cases}$$



Therefore,

$$\begin{aligned} \measuredangle H(s_0) &= r\pi \qquad r: \text{ total \# of poles and zeros to the right of } s_0 \\ &= \pi \qquad \text{if } r \text{ is odd.} \end{aligned}$$

4) Branches between two real poles must break away into the complex plane for some $K > 0$. The break-away and break-in points can be determined by solving for the roots of

$$\frac{dH(s)}{ds} = 0$$

that lie on the real line.

Example 2 above:

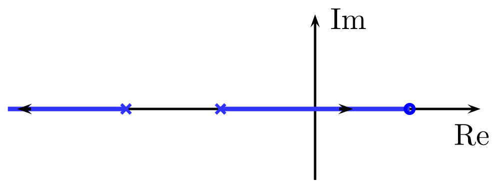
$$H(s) = \frac{1}{Ms^2 + bs}$$

$$\frac{dH}{ds} = \frac{-2Ms - b}{(Ms^2 + bs)^2} = 0 \quad \Rightarrow \quad s = \frac{-b}{2M}$$

Example 3:

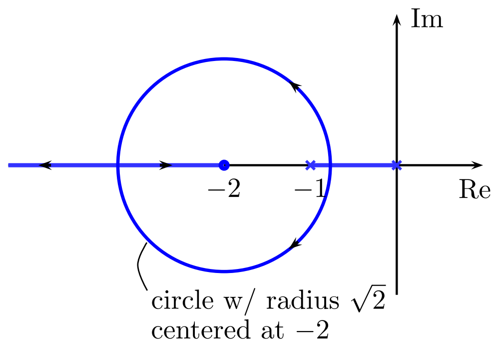
$$H(s) = \frac{s-1}{(s+1)(s+2)}$$

$\underbrace{n = 2, \; m = 1}_{\text{one asymptote with angle } 180^o}$, zeros: $s = 1$, poles: $s = 1, -2$.



Example 4:

$$H(s) = \frac{s+2}{s(s+1)} \quad n - m = 1 \text{ asymptote with angle } 180^o$$



Break-away/ break-in points:

$$\frac{dH}{ds} = \frac{s^2 + s - (2s+1)(s+2)}{s^2(s+1)^2} = 0$$

$$s^2 + s - (2s^2 + 5s + 2) = 0$$

$$s^2 + 4s + 2 = 0 \Rightarrow s = \frac{-4 \mp \sqrt{8}}{2} = -2 \mp \sqrt{2}$$

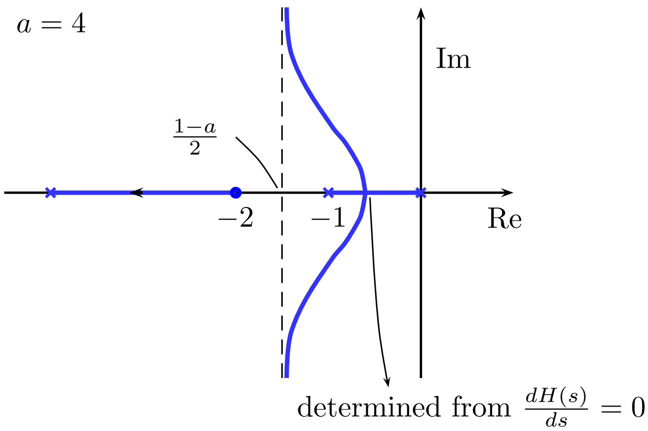
Example 5:

$$H(s) = \frac{s+2}{s(s+1)(s+a)} \quad a > 2$$

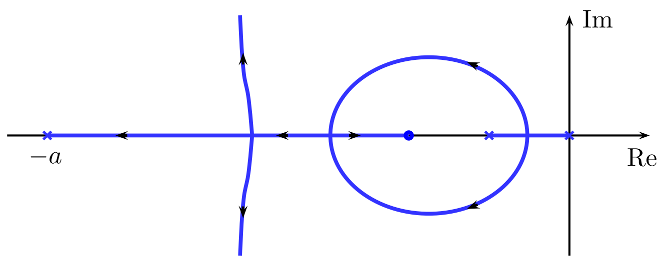
(pole at $-a$ added to the previous example)

$n - m = 2$, therefore two asymptotes with angles $\mp 90^o$

center of the asymptotes: $\frac{(0-1-a)-(-2)}{2} = \frac{1-a}{2}$

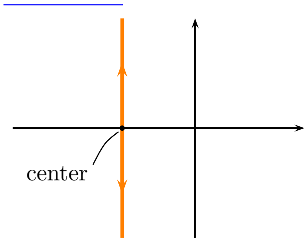


For large enough a, $\frac{dH}{ds} = 0$ has three real, negative roots:

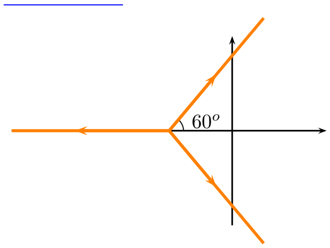


<u>High-Gain Instability:</u>
Large feedback gain causes instability if:
1) $H(s)$ has zeros in the right-half plane
2) $n - m \geq 3$

<u>$n - m = 2$</u>

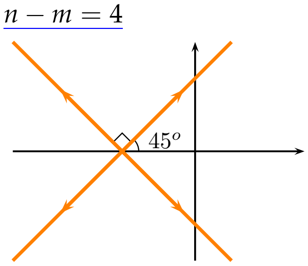


<u>$n - m = 3$</u>

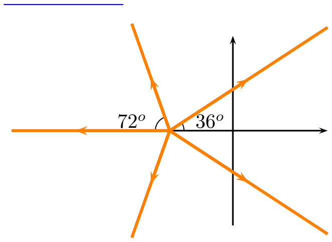


stable but poorly damped as $K \nearrow$

<u>$n - m = 4$</u>

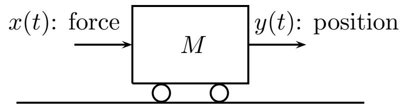


<u>$n - m = 5$</u>



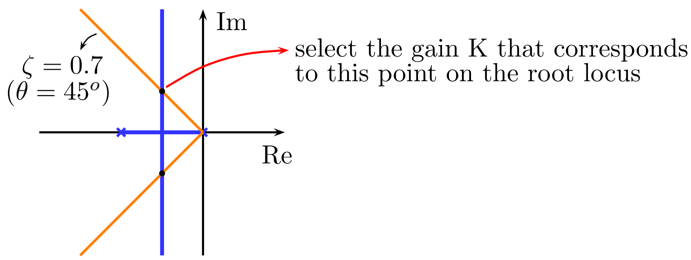
Control Design by Root Locus

<u>Example</u>:

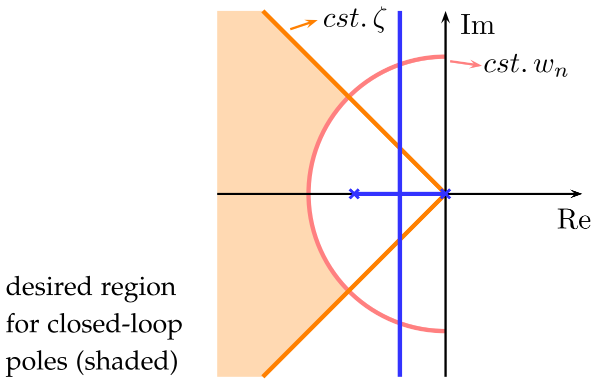


$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} = x(t) \quad \rightarrow \quad H_p(s) = \frac{1}{Ms^2 + bs}$$

Suppose a damping ratio of $\zeta = 0.7$ is desired:



Suppose, in addition to ζ, a lower bound on ω_n is specified:



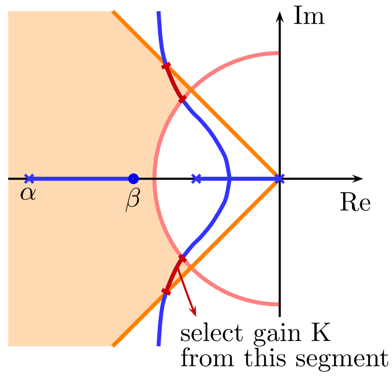
The root locus doesn't go through the desired region, therefore constant gain control won't work. Try the controller:

$$H_c(s) = K\frac{s-\beta}{s-\alpha} \quad \alpha < \beta < 0 \ \ \text{(pole to the left of zero)}$$

Closed-loop poles:

$$1 + \underbrace{\overbrace{K\frac{s-\beta}{s-\alpha}}^{H_c(s)}\overbrace{\frac{1}{s(Ms+b)}}^{H_p(s)}}_{H(s)} = 0$$

Select α, β such that the root locus passes through the desired region



A controller of the form

$$\boxed{H_c(s) = K\frac{s-\beta}{s-\alpha} \quad \alpha < \beta < 0}$$

is called a "lead controller".

<u>Example:</u>

$$\begin{aligned} H_c(s) &= \frac{s+1}{s+10} \\ H_c(j\omega) &= \frac{1}{10}\frac{1+j\omega}{1+j\omega/10} \end{aligned}$$

$$20log_{10}|H_c(j\omega)| = -20 - 20log_{10}|1+j\omega/10| + 20log_{10}|1+j\omega|$$